

LARGE OUTGOING SOLUTIONS TO SUPERCRITICAL WAVE EQUATIONS

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ABSTRACT. We prove the existence of global solutions to the energy-supercritical wave equation in \mathbb{R}^{3+1}

$$u_{tt} - \Delta u + |u|^N u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1, \quad 4 < N < \infty,$$

for a large class of radially symmetric finite-energy initial data.

Functions in this class are characterized as being outgoing under the linear flow — for a specific meaning of “outgoing” defined below.

In particular, we construct global solutions for initial data with large (even infinite) critical Sobolev norm and large critical Lebesgue norm.

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1. INTRODUCTION

1.1. Statement of the main results. Consider the semilinear wave equation in \mathbb{R}^{3+1}

$$u_{tt} - \Delta u \pm |u|^N u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (1.1)$$

The equation is called focusing or defocusing according to whether the sign of the nonlinearity is $-$ or $+$.

For $\alpha \in (0, \infty)$, this equation is invariant under the scaling symmetries

$$u(x, t) \mapsto \alpha^{2/N} u(\alpha x, \alpha t),$$

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as well as under the Lorentz group of transformations. Restricted to the initial data, the rescaling is

$$(u_0(x), u_1(x)) \mapsto (\alpha^{2/N} u_0(\alpha x), \alpha^{1+2/N} u_1(\alpha x)). \quad (1.2)$$

For $s_c = 3/2 - 2/N$, the $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ Sobolev norm is invariant under the rescaling (1.2), making it the critical Sobolev norm for the equation. Equation (1.1) is locally well-posed in the $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm. Note that the corresponding (critical) Lebesgue norm is $u_0 \in L^{p_c}$ with $p_c = 3N/2$.

All non-critical norms of the solution can be made arbitrarily large or small by rescaling, but critical norms remain constant after rescaling.

An important conserved quantity for equation (1.1) is energy, defined as

$$E[u] := \int_{\mathbb{R}^3 \times \{t\}} \frac{1}{2} |u_t(x, t)|^2 + \frac{1}{2} |\nabla u(x, t)|^2 \pm \frac{1}{N+2} |u(x, t)|^{N+2} dx.$$

In case the equation is defocusing, energy controls the $\dot{H}^1 \times L^2$ norm of the solution, also called the energy norm.

Equation (1.1) is energy-supercritical (or, in brief, supercritical) if $N > 4$. The difficulty of the initial-value problem in this case lies in the fact that solutions cannot be controlled in the energy norm (as $s_c > 1$) and no higher-level conserved quantities can be used either.

By the standard local existence theory, based on Strichartz estimates, any initial data in the critical Sobolev space $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ produces a solution, locally in time. If the initial data is sufficiently small in the critical Sobolev norm, then the corresponding solution exists globally in time and disperses, meaning that, for example, it has finite $L_{t,x}^{2N}$ norm (the endpoints are $L_t^\infty L_x^{3N/2}$, which is not dispersive, and $L_t^{N/2} L_x^\infty$, which is achieved for $N > 4$ or $N = 4$ and radially symmetric solutions).

In this paper we only consider the case of radially symmetric, i.e. rotation-invariant, solutions. We also assume all solutions are real-valued.

For simplicity we suppose that N is an integer in (1.1). This makes little actual difference in the proof.

Our first result is an existence result for the class of initial data

$$((\dot{H}^1 \cap L^\infty) \times L^2)_{out} + \dot{H}^s \times \dot{H}^{s-1} := \{(u_0, u_1) = (v_0, v_1) + (w_0, w_1) \mid \\ (v_0, v_1) \in ((\dot{H}^1 \cap L^\infty) \times L^2)_{out}, (w_0, w_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}\},$$

where $((\dot{H}^1 \cap L^\infty) \times L^2)_{out}$ means radial and outgoing following Definition 3.5. Note that for data in this class the outgoing component is in a weaker space than $\dot{H}^s \times \dot{H}^{s-1}$, but the incoming component must still be in $\dot{H}^s \times \dot{H}^{s-1}$.

Theorem 1.1. *Assume that $N \in (4, 12]$, the initial data $(u_0, u_1) = (v_0, v_1) + (w_0, w_1)$ decompose into a radial and outgoing component (v_0, v_1) and a second radial component $(w_0, w_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ such that*

$$\|v_0\|_{\dot{H}^1}^{4/N} \|v_0\|_{L^\infty}^{1-4/N} + \|(w_0, w_1)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} << 1$$

is sufficiently small. Then the corresponding solution u to (1.1) exists globally, forward in time, remains small in $((\dot{H}_x^1 \cap L_x^\infty) \times L_x^2)_{out} + \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$, and disperses:

$$\|u\|_{L_t^{N/2} L_x^\infty} \lesssim \|v_0\|_{\dot{H}^1}^{4/N} \|v_0\|_{L^\infty}^{1-4/N} + \|(w_0, w_1)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}. \quad (1.3)$$

In addition u scatters: there exist $(w_{0+}, w_{1+}) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ such that

$$\lim_{t \rightarrow \infty} \|(u(t), u_t(t)) - \Phi(t)(v_0, v_1) - \Phi(t)(w_{0+}, w_{1+})\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} = 0. \quad (1.4)$$

Here $\Phi(t)$ is the flow induced by the linear wave equation.

If $(v_0, v_1) \in ((\dot{H}^1 \cap L^\infty) \times L^2)_{out}$ and $(w_0, w_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ are not small, then there exist an interval $I = [0, T]$ with $T > 0$ and a solution u to (1.1) defined on $\mathbb{R}^3 \times I$, with (u_0, u_1) as initial data, such that $(u(t), u_t(t)) \in ((\dot{H}_x^1 \cap L_x^\infty) \times L_x^2)_{out} + \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ for $t \in I$ and

$$\|u\|_{L_t^{N/2} L_x^\infty(\mathbb{R}^3 \times I)} < \infty.$$

The case $N = 4$ corresponds to $s_c = 1$ and $N = 12$ corresponds to $s_c = 4/3$. The conclusion is still true, but trivial, when $N = 4$.

Thus, we obtain global dispersive and scattering solutions for initial data of arbitrarily high, even infinite critical Sobolev norms. On the other hand, these initial data are still small in the critical L^{p_c} norm.

Dropping the scaling invariance, we can obtain a local existence result for large $((\dot{H}^1 \cap L^\infty) \times L^2)_{out}$ initial data in the subcritical sense (i.e. where the time of existence only depends on the size of the initial data). We can also obtain a global existence result for small initial data, such that the solution remains bounded in $((\dot{H}^1 \cap L^\infty) \times L^2)_{out} + \dot{H}^2 \cap \dot{H}^1 \times \dot{H}^1 \cap L^2$ for all times, i.e. the incoming component of the solution gains a full derivative. See Proposition 5.2 for both results.

As a consequence of Theorem 1.1, outgoing and radial finite energy initial data of any size lead to a global solution forward in time if they are supported sufficiently far away from the origin.

Corollary 1.2. *Assume that $N \in (4, 12]$, the initial data (u_0, u_1) are radial, outgoing according to Definition 3.5, supported outside the sphere $B(0, R)$, and*

$$\|u_0\|_{\dot{H}^1}^2 R^{4/N-1} \ll 1 \quad (1.5)$$

is sufficiently small. Then the corresponding solution u to (1.1) exists globally, forward in time, and disperses, having finite $L_t^{N/2} L_x^\infty$ norm.

Again, one can add a small $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ perturbation to the initial data, either outgoing or incoming, without changing the result. In addition, note that the conclusion is still true, but trivial, when $N = 4$.

Remark 1.3. Note that condition (1.5) becomes automatically satisfied (up to a small error) if the equation (1.1) is defocusing and we wait for long enough. This is because the $\dot{H}^1 \times L^2$ energy norm remains bounded by

the energy $E[u]$, hence the left-hand side of (1.5) improves with time: the solution becomes more outgoing and gets further removed from the origin, through dispersion. Since $4/N - 1 < 0$, $R^{4/N-1} \rightarrow 0$ as $R \rightarrow \infty$.

Thus, our results could be part of the the process of showing global in time existence and scattering for any large radial solution to (1.1), after the solution is first shown to exist for a sufficiently long, but finite time.

The next result shows that it is not necessary to assume that the initial data have finite energy — bounded and of compact support will suffice.

Theorem 1.4. *Assume that $N \in [4, \infty)$ and (u_0, u_1) are radial initial data, outgoing according to Definition 3.5, such that u_0 is bounded and supported on $B(0, R)$. Then, as long as*

$$\|u_0\|_{L^\infty} R^{2/N} \ll 1$$

is sufficiently small, the corresponding solution u to (1.1) exists globally, forward in time, and disperses:

$$\|u\|_{L_x^{3N/2} L_t^\infty \cap L_x^\infty L_t^{N/2}} \lesssim \|u_0\|_{L^\infty} R^{2/N}.$$

In particular, these solutions have finite homogenous $L_{t,x}^{2N}$ norm.

Note that these initial data can be arbitrarily large in the critical Sobolev norm, but must still be small in the critical L^{p_c} norm.

Again, it is possible to add a small $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ perturbation, either incoming or outgoing, to the initial data.

It is easy to obtain a local existence result for large radial outgoing initial data (u_0, u_1) when $u_0 \in L^\infty$, see Proposition 5.1.

Finally, it is possible to construct global solutions to (1.1) for initial data with arbitrarily high critical L^{p_c} norm. For simplicity we assume that N is an even integer.

Theorem 1.5 (Main result). *Assume that $N \in (2, \infty)$ is even. For any $L > 0$ there exist initial data (u_0, u_1) such that $\|u_0\|_{L^{p_c}} \geq L$, the corresponding solution u of (1.1) is global, forward in time, and*

$$\|u\|_{\langle x \rangle^{-1} L_{t,x}^\infty \cap \langle x \rangle^{-1} L_x^\infty L_t^1} < \infty.$$

This implies that the solutions have finite critical $L_{t,x}^{2N}$ norm.

Note that these initial data necessarily also have arbitrarily high \dot{H}^{s_c} norms, but we already had such examples from Theorem 1.1.

From our construction it follows that $\|u_0\|_{L^\infty(|x| \geq 1)} \gg 1$, see [KrSc] for a similar result obtained by completely different methods. Our construction is based on taking outgoing initial data concentrated on a thin spherical shell.

All these results hold in both the focusing and the defocusing case, regardless of the sign of the nonlinearity in (1.1).

Also note that if we assume the initial data are smooth then the solution is also smooth.

1.2. History of the problem. The first well-posedness result for large data supercritical problems was obtained by Tao [Tao], for the logarithmically supercritical defocusing wave equation that he introduced

$$u_{tt} - \Delta u + u^5 \log(2 + u^2) = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (1.6)$$

[Tao] proved global well-posedness and scattering for radial initial data. The starting point of [Tao] was an observation made in [GSV] for the energy-critical problem.

Further results in the supercritical case belong to Roy [Roy1] [Roy2], who proved the scattering of solutions to the log-log-supercritical equation

$$u_{tt} - \Delta u + u^5 \log^c(\log(10 + u^2)) = 0, \quad u(0) = u_0, \quad u_t(0) = u_1,$$

$0 < c < \frac{8}{225}$, without the radial assumption.

Another supercritical result belongs to Struwe [Str], who proved the global well-posedness of the supercritical equation

$$u_{tt} - \Delta u + u e^{u^2} = 0, \quad u(0) = u_0, \quad u_t(0) = u_1,$$

(supercritical in the case when $E[u] > 2\pi$) for arbitrary radial smooth initial data.

A different series of results asserts the conditional well-posedness of supercritical equations. If the critical Sobolev norm $\|(u, u_t)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}$ of a solution to (1.1) stays bounded, then the solution exists globally and disperses. Such findings belong to Killip–Visan [KiVi1], [KiVi2] and Bulut [Bul1], [Bul2], [Bul3] in the defocusing case and Duyckaerts–Kenig–Merle [DKM] in the focusing case.

All these conditional results are based on methods developed by Bourgain [Bou], Colliander–Keel–Staffilani–Takaoka–Tao [CKSTT], Kenig–Merle [KeMe], and Keraani [Ker] in the energy-critical case.

Another recent result belongs to Kriger–Schlag [KrSc], who construct large initial data solutions to the supercritical equation (1.1).

The new results presented in this paper do not require the solution to be bounded or even finite in the critical Sobolev norm $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$. Indeed, this critical norm can be replaced with the $\dot{H}^1 \times L^2$ energy norm of outgoing initial data, provided they are supported sufficiently far out (or the L^∞ norm is sufficiently small).

Notably, any smooth and radial scattering solution of equation (1.1) can be shown to fulfill the conditions of Corollary 1.2 after sufficient time has elapsed. The problem, then, rests in controlling the solution on a sufficiently long, but finite time interval. This can be achieved numerically, for example.

The energy norm being finite is also not necessary, as we construct solutions for bounded and compactly supported initial data. A variant of our construction (Theorem 1.5) leads to initial data that are large in the sense of [KrSc], i.e. $\|u_0\|_{L^\infty(|x| \geq 1)} \gg 1$.

Our result and the result of [KrSc] are rather different. Our result is based on multilinear estimates and [KrSc] is based on a nonlinear construction.

In addition, the solution of [KrSc] only logarithmically fails to be in the critical Sobolev space and in fact is bounded in a critical Lorentz space (i.e. $L^{3N/2,\infty}$). By contrast, our solutions can fail to be in the critical Sobolev space by a wide margin and are large in the critical Lebesgue norm.

As an aside, we also prove a global well-posedness result for a wide class of large initial data for the focusing equation (1.1). The solutions we construct satisfy the equation in a rather weak sense, being possibly infinite.

The decomposition into outgoing and incoming states which is the basis of our paper is not entirely new. In fact, the incoming condition in our paper resembles a condition from Engquist–Majda [EnMa], see formula (1.27) in that paper.

We expect the same method to lead to an improvement in the energy-critical and subcritical cases, by allowing us to prove, for example, global well-posedness for $(\dot{H}^{1/2} \cap L^\infty) \times \dot{H}^{-1/2}$ outgoing initial data, thus requiring fewer derivatives than the critical Sobolev exponent for the equation. It may also be possible to obtain results in the $\dot{H}^{1/2}$ -subcritical range. All these improvements will be explored in a future paper.

Another expected result is the well-posedness of equation (1.1) for arbitrary large initial data, after projecting the nonlinearity on the outgoing states. This will constitute the subject of another paper.

This paper is organized as follows: in Section 3 we state several results about incoming and outgoing states for the linear flow, in Section 4 we enounce some standard existence results, and in Section 5 we prove the theorems stated in the introduction. In the Appendix we state a large data global well-posedness result (which holds in the supercritical case), not related to the rest of the paper.

2. NOTATIONS

$A \lesssim B$ means that $|A| \leq C|B|$ for some constant C . We denote various constants, not always the same, by C .

The Laplacian is the operator on \mathbb{R}^3 $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

We denote by L^p the Lebesgue spaces, by \dot{H}^s and $\dot{W}^{s,p}$ (fractional) homogeneous Sobolev spaces, and by $L^{p,q}$ Lorentz spaces.

H^s are Hilbert spaces and so is $\dot{H}^1 \times L^2$, under the norm

$$\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} = (\|u_0\|_{\dot{H}^1}^2 + \|u_1\|_{L^2}^2)^{1/2}.$$

By \dot{H}_{rad}^s , etc. we designate the radial version of these spaces. For a radially symmetric function $u(x)$, we let $u(r) := u(x)$ for $|x| = r$.

By $(\dot{H}^1 \times L^2)_{out}$ we mean the space of outgoing radially symmetric $\dot{H}^1 \times L^2$ initial data, see Definition 3.5.

We define the mixed-norm spaces on $\mathbb{R}^3 \times [0, \infty)$

$$L_t^p L_x^q := \left\{ f \mid \|f\|_{L_t^p L_x^q} := \left(\int_0^\infty \|f(x, t)\|_{L_x^q}^p dt \right)^{1/p} < \infty \right\},$$

with the standard modification for $p = \infty$, and likewise for the reversed mixed-norm spaces $L_x^q L_t^p$. We use a similar definition for $L_t^p \dot{W}_x^{s,p}$. Also, for $I \subset [0, \infty)$, let $\|f\|_{L_t^p L_x^q(\mathbb{R}^3 \times I)} := \|\chi_I(t)f\|_{L_t^p L_x^q}$, where χ_I is the characteristic function of I .

We also denote $B(0, R) := \{x \in \mathbb{R}^3 \mid |x| \leq R\}$.

Let D be the Fourier multiplier $|\xi|$, δ_0 be Dirac's delta at zero, and χ denote the characteristic function of a set.

Let $\Phi(t) : \dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2$ be the flow of the linear wave equation in three dimensions: for

$$u_{tt} - \Delta u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1,$$

we set $\Phi(t)(u_0, u_1) = (\Phi_0(t)(u_0, u_1), \Phi_1(t)(u_0, u_1)) := (u(t), u_t(t))$.

Also let $\phi(t) : L^2 \times \dot{H}^{-1} \rightarrow L^2 \times \dot{H}^{-1}$ be the flow of the linear wave equation in dimension one (on a half-line with Neumann boundary conditions): for

$$v_{tt} - v_{rr} = 0, \quad v(0) = v_0, \quad v_t(0) = v_1, \quad v_r(0, t) = 0,$$

we set $\phi(t)(v_0, v_1) := (v(t), v_t(t))$.

In this paper we only consider mild solutions to (1.1), i.e. solutions to the following equivalent integral equation:

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 \mp \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}|u(s)|^N u(s) ds.$$

3. OUTGOING AND INCOMING STATES FOR THE FREE FLOW

In order to define outgoing and incoming states for the linear wave equation, we reduce the equation to a one-dimensional problem, where identifying such states is straightforward.

Namely, we write each radial function $u(r = |x|)$ on \mathbb{R}^3 as a superposition of identical one-coordinate functions in every possible direction. For each radial function $u(x)$ on \mathbb{R}^3 there exists a function $T(u)(s)$ defined on \mathbb{R} such that

$$u(x) = \int_{S^2} T(u)(x \cdot \omega) d\omega. \quad (3.1)$$

Taking $T(u)$ to be supported on $[0, \infty)$, this works out to

Lemma 3.1. *For radial $u(x) = u(|x|)$ and $T(u)$ related by (3.1), such that $T(u)$ is supported on $[0, \infty)$,*

$$T(u)(r) = \frac{1}{2\pi}(ru(r))', \quad u(r) = \frac{2\pi}{r} \int_0^r T(u)(s) ds. \quad (3.2)$$

Proof of Lemma 3.1. With no loss of generality assume that $x = (0, 0, r)$ and write ω in polar coordinates as $\omega = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Then

$$u(r) = \int_0^\pi \int_0^{2\pi} T(u)(r \cos \theta) \sin \theta d\phi d\theta = \frac{2\pi}{r} \int_{-r}^r T(u)(s) ds = \frac{2\pi}{r} \int_0^r T(u)(s) ds.$$

□

Conveniently, T is a constant times a unitary map from L_{rad}^2 (the space of radial L^2 functions) to $\dot{H}^{-1}([0, \infty))$ and bounded from \dot{H}_{rad}^1 to $L^2([0, \infty))$ — the latter by Hardy's inequality. The inverse operator T^{-1} is also bounded from $L^2([0, \infty))$ to \dot{H}_{rad}^1 .

Lemma 3.2. *With T defined by (3.1), $\|T(u)\|_{\dot{H}^{-1}([0, \infty))} = \frac{1}{\sqrt{\pi}}\|u\|_{L_{rad}^2}$. Moreover, $\|T(u)\|_{L^2([0, \infty))} \lesssim \|u\|_{\dot{H}_{rad}^1}$ and $\|u\|_{\dot{H}_{rad}^1} \lesssim \|T(u)\|_{L^2([0, \infty))}$.*

Remark 3.3. This shows that $\|u\| := \|T(u)\|_{L^2([0, \infty))} = \|(ru(r))'\|_{L^2([0, \infty))}$ is another norm on \dot{H}_{rad}^1 equivalent to the usual one.

Proof of Lemma 3.2. Note that $u \in L_{rad}^2$ if and only if

$$\|u\|_{L_{rad}^2}^2 = 4\pi \int_0^\infty |u(r)|^2 r^2 dr.$$

In particular, $\|u\|_{L_{rad}^2} = 2\sqrt{\pi}\|ru(r)\|_{L^2([0, \infty))} = 2\sqrt{\pi}\|(ru(r))'\|_{\dot{H}^{-1}([0, \infty))}$.

Next, note that by Hardy's inequality

$$\|u/r\|_{L_{rad}^2}^2 = 4\pi \int_0^\infty |u(r)|^2 dr \lesssim \|u\|_{\dot{H}_{rad}^1}^2 = 4\pi \int_0^\infty |u_r(r)|^2 r^2 dr.$$

Therefore $\|(ru(r))'\|_{L^2([0, \infty))} \leq \|u\|_{L_{rad}^2} + \|ru'(r)\|_{L^2([0, \infty))} \lesssim \|u\|_{\dot{H}_{rad}^1}$.

Finally, let $T(u) = v$. Then by (3.2)

$$\|u\|_{\dot{H}_{rad}^1}^2 = 4\pi \int_0^\infty |u_r(r)|^2 r^2 dr \lesssim \int_0^\infty |v(r)|^2 dr + \int_0^\infty \frac{1}{r^2} \left(\int_0^r v(s) ds \right)^2 dr.$$

Then, $\frac{1}{r} \int_0^r v(s) ds = \int_0^1 v(\alpha r) d\alpha$ and $\|v(\alpha \cdot)\|_{L^2} = \alpha^{-1/2}\|v\|_{L^2}$, so

$$\left\| \int_0^1 v(\alpha r) d\alpha \right\|_{L_r^2} \leq \int_0^1 \|v(\alpha \cdot)\|_{L^2} d\alpha = \|v\|_{L^2} \int_0^1 \alpha^{-1/2} d\alpha \lesssim \|v\|_{L^2}.$$

This proves the last statement of the lemma. \square

For non-radial functions, a similar decomposition into one-coordinate functions can be obtained, but the computation is more complicated. One restricts the three-dimensional Fourier transform along each line through the origin, then takes the inverse one-dimensional Fourier transform. This is related to the Radon transform.

So far, the transformation T is completely general; however, the subsequent computation is not and will be different for, say, Schrödinger's equation.

Lemma 3.4. *There exist bounded operators P_+ and P_- on $\dot{H}_{rad}^1 \times L_{rad}^2$, given by*

$$P_+(u_0, u_1) = \left(\frac{1}{2} \left(u_0 - \frac{1}{r} \int_0^r s u_1(s) ds \right), \frac{1}{2} \left(-(u_0)_r - \frac{u_0}{r} + u_1 \right) \right) \quad (3.3)$$

and

$$P_-(u_0, u_1) = \left(\frac{1}{2} \left(u_0 + \frac{1}{r} \int_0^r s u_1(s) ds \right), \frac{1}{2} \left((u_0)_r + \frac{u_0}{r} + u_1 \right) \right), \quad (3.4)$$

such that $I = P_+ + P_-$, $P_+^2 = P_+$, and $P_-^2 = P_-$.

If $\Phi(t)$ is the flow of the linear equation then for $t \geq 0$ $P_- \Phi(t) P_+ = 0$ and for $t \leq 0$ $P_+ \Phi(t) P_- = 0$. In addition, for $t \geq 0$ $\Phi(t) P_+(u_0, u_1)$ is supported on $\overline{\mathbb{R}^3 \setminus B(0, t)}$ and for $t \leq 0$ $\Phi(t) P_-(u_0, u_1)$ is supported on $\overline{\mathbb{R}^3 \setminus B(0, -t)}$.

Definition 3.5. P_+ and P_- are called the projection on outgoing, respectively incoming states. We call any radial (u_0, u_1) such that $P_-(u_0, u_1) = 0$ outgoing; if $P_+(u_0, u_1) = 0$ we call it incoming.

Remark 3.6. P_+ and P_- are self-adjoint on $\dot{H}_{rad}^1 \times L_{rad}^2$ with the norm $\|(u_0, u_1)\| := (\|(ru_0(r))'\|_{L^2([0, \infty))}^2 + \|ru_1(r)\|_{L^2([0, \infty))}^2)^{1/2}$, see Remark 3.3.

Proof. Given a radial solution u of the free wave equation

$$u_{tt} - \Delta u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1, \quad (3.5)$$

each of the essentially one-dimensional functions $T(u(t))(x \cdot \omega)$ fulfills a one-dimensional wave equation of the form

$$v_{tt} - v_{rr} = 0, \quad v(0) = v_0, \quad v_t(0) = v_1, \quad (3.6)$$

where

$$v_0 = \frac{1}{2\pi}(ru_0(r))', \quad v_1 = \frac{1}{2\pi}(ru_1(r))'.$$

Indeed, in radial coordinates one has that $u_{tt} - u_{rr} - (2/r)u_r = 0$ or equivalently that $(ru)_{tt} - (ru)_{rr} = 0$. Since $v = T(u) = \frac{1}{2\pi}(ru(r))'$, taking a derivative leads to (3.6), which holds in the weak sense.

To fix ideas we assume that $(u_0, u_1) \in \dot{H}_{rad}^1 \times L_{rad}^2$, so $(v_0, v_1) \in L^2 \times \dot{H}^{-1}$ by Lemma 3.2. Note that v_0 and v_1 are supported on $[0, \infty)$.

In addition, we consider equation (3.6) on a half-line only and impose the Neumann boundary condition $v_r(0, t) = 0$. This is justified because for a smooth radial solution u one necessarily has $u_r(0, t) = 0$ and $v_r(0, t) = \frac{1}{2\pi}(ru)_{rr}(0, t) = \frac{1}{\pi}u_r(0, t)$.

The Neumann boundary condition means that the solution is reflected back at the boundary or, in other words, that it could be extended by symmetry to the negative half-axis.

Equation (3.6) then has solutions of the form

$$v(r, t) = \chi_{r \geq t} v_+(r - t) + \chi_{r \leq t} v_-(t - r) + \chi_{r+t \geq 0} v_-(r + t) + \chi_{r+t \leq 0} v_+(-r - t),$$

where by d'Alembert's formula

$$v_+(r) = \frac{1}{2}(v_0(r) - \partial_r^{-1} v_1(r)), \quad v_-(r) = \frac{1}{2}(v_0(r) + \partial_r^{-1} v_1(r)). \quad (3.7)$$

Here ∂_r^{-1} denotes the unique antiderivative of a \dot{H}^{-1} distribution that belongs to L^2 . Note that since v_1 is supported on $[0, \infty)$, $\partial_r^{-1} v_1$ is also supported on $[0, \infty)$ (in fact it is given by $\frac{1}{2\pi}ru_1(r)$).

Thus both v_+ and v_- are supported on $[0, \infty)$.

At time t , the outgoing component of v , which moves in the positive direction with velocity 1, consists of

$$v_{out}(r, t) = \chi_{r \geq t} v_+(r - t) + \chi_{r \leq t} v_-(t - r).$$

The incoming component consists of

$$v_{in}(r, t) = \chi_{r+t \geq 0} v_-(r+t) + \chi_{r+t \leq 0} v_+(-r-t).$$

In particular, at $t = 0$ the outgoing component is v_+ and the incoming component is v_- .

Note that as t grows the incoming component hits the origin and becomes outgoing. If we wait for long enough, most of the solution becomes outgoing. Conversely, if we reverse time flow, as $t \rightarrow -\infty$ all of the solution becomes incoming.

In order to obtain a general formula for the outgoing projection, without loss of generality we restrict our attention to time 0. Let $\pi_+(v_0, v_1)$ be the initial data of the outgoing component, i.e.

$$\pi_+(v_0, v_1)(r) := (v_+(r-t) |_{t=0}, \partial_t(v_+(r-t)) |_{t=0}) = (v_+(r), -v'_+(r))$$

and likewise

$$\pi_-(v_0, v_1)(r) := (v_-(r+t) |_{t=0}, \partial_t(v_-(r+t)) |_{t=0}) = (v_-(r), v'_-(r)).$$

Note that $\pi_+(v_0, v_1)$ is the initial data for the solution $v_+(r-t)$ of equation (3.6) on the time interval $[0, \infty)$ (which moves with velocity 1 in the positive direction) and same for π_- on $(-\infty, 0]$.

By d'Alembert's formula (3.7), π_+ then has the form

$$\pi_+(v_0, v_1)(r) := \left(\frac{1}{2}(v_0(r) - \partial_r^{-1}v_1(r)), \frac{1}{2}(-v'_0(r) + v_1(r)) \right)$$

and the incoming component π_- has the form

$$\pi_-(v_0, v_1)(r) := \left(\frac{1}{2}(v_0(r) + \partial_r^{-1}v_1(r)), \frac{1}{2}(v'_0(r) + v_1(r)) \right).$$

Note that $\pi_+ + \pi_- = I$, $\pi_+^2 = \pi_+$, $\pi_-^2 = \pi_-$, and $\pi_+\pi_- = \pi_-\pi_+ = 0$. In particular, note that $\partial_r^{-1}v'_0 = v_0$, because in any case an antiderivative of v'_0 must be of the form $v_0 + c$ and this is in L^2 only if $c = 0$.

Since by definition $\langle u, v \rangle_{L^2} = \langle u', v' \rangle_{\dot{H}^{-1}}$, a simple computation shows that π_+ and π_- are bounded, self-adjoint operators on $L^2 \times \dot{H}^{-1}$.

Thus, both π_+ and π_- are orthogonal projections, in a proper setting (on $L^2 \times \dot{H}^{-1}$ or more generally on $\dot{H}^s \times \dot{H}^{s-1}$).

Note that a solution of the form $v := v_+(r-t)$ preserves the property that $\partial_t v = -\partial_r v$ and hence that $\pi_-(v(t), v_t(t)) = 0$ for all $t \geq 0$. In other words, if we denote by $\phi(t)$ the flow induced by the linear equation (3.6) on $L^2 \times \dot{H}^{-1}$, then for $t \geq 0$

$$\pi_-\phi(t)\pi_+(v_0, v_1) = 0.$$

Furthermore, in this case $\phi(t)\pi_+(v_0, v_1)$ is obviously supported on $[t, \infty)$.

Likewise, for $t \leq 0$ $\pi_+\phi(t)\pi_-(v_0, v_1) = 0$ and $\text{supp } \phi(t)\pi_-(v_0, v_1) \subset [-t, \infty)$.

The outgoing component of u corresponds to the outgoing component $\pi_+(v, v_t)$ of v traveling in the positive direction. Conjugating the projections π_+ and π_- by the transformation T defined by (3.1), we obtain the corresponding operators for radial functions in \mathbb{R}^3 . Letting $P_+ := T^{-1}\pi_+T$,

$P_- := T^{-1}\pi_-T$, we obtain formulas (3.3) and (3.4). Both operators are bounded on $\dot{H}_{rad}^1 \times L_{rad}^2$ due to Lemma 3.2.

As an easy consequence of the properties of π_+ and π_- , we get that $P_+ + P_- = I$, $P_+P_- = 0$, $P_+^2 = P_+$, $P_-^2 = P_-$, and all the other stated properties of P_+ and P_- . \square

We now prove some properties of the nonlocal operator that appears in the definition of the projections on incoming and outgoing states. This leads among other things to the boundedness of the projections on outgoing and incoming states. Also note that if the initial data (u_0, u_1) are purely outgoing or are purely incoming, then u_0 determines u_1 and vice-versa. Furthermore, this can be made into a quantitative estimate.

Lemma 3.7. *For radial $f \in L^2$*

$$\left\| \frac{1}{r} \int_0^r \rho f(\rho) d\rho \right\|_{\dot{H}_{rad}^1} \lesssim \|f\|_{L_{rad}^2}.$$

More generally, for $0 \leq s < 3/2$

$$\left\| \frac{1}{r} \int_0^r \rho f(\rho) d\rho \right\|_{\dot{H}_{rad}^{s+1}} \lesssim \|f\|_{\dot{H}_{rad}^s}. \quad (3.8)$$

Consequently, P_+ and P_- are bounded on $\dot{H}^s \times \dot{H}^{s-1}$ for $1 \leq s < 3/2$.

Furthermore, if (u_0, u_1) are purely outgoing or purely incoming, then $\|u_0\|_{\dot{H}^s} \sim \|u_1\|_{\dot{H}^{s-1}}$ for $1 \leq s < 3/2$.

Proof of Lemma 3.7. By differentiation we see that

$$\left\| \frac{1}{r} \int_0^r \rho f(\rho) d\rho \right\|_{\dot{H}_{rad}^1} = \left\| f - \frac{1}{r^2} \int_0^r \rho f(\rho) d\rho \right\|_{L_{rad}^2}$$

and then

$$\begin{aligned} \left\| \frac{1}{r^2} \int_0^r \rho f(\rho) d\rho \right\|_{L_{rad}^2} &\lesssim \left\| \frac{1}{r} \int_0^r \rho f(\rho) d\rho \right\|_{L_r^2([0, \infty))} = \left\| \int_0^1 \alpha f(\alpha r) d\alpha \right\|_{L_r^2([0, \infty))} \\ &\lesssim \int_0^1 \|\alpha r f(\alpha r)\|_{L_r^2([0, \infty))} d\alpha = \|rf(r)\|_{L_r^2([0, \infty))} \int_0^1 \alpha^{-1/2} d\alpha \lesssim \|f\|_{L_{rad}^2}. \end{aligned} \quad (3.9)$$

By the same reasoning, (3.8) follows for $0 \leq s \leq 1$ from

$$\left\| \frac{1}{r^2} \int_0^r \rho f(\rho) d\rho \right\|_{\dot{H}_{rad}^s} \lesssim \|f\|_{\dot{H}_{rad}^s}.$$

This in turn follows by interpolation between the $s = 0$ case proved above (see 3.9) and the $s = 1$ case, which is implied by Hardy's inequality $\|f/|x|\|_{L^2} \lesssim \|f\|_{\dot{H}^1}$ and

$$\begin{aligned} \left\| \frac{1}{r^3} \int_0^r \rho f(\rho) d\rho \right\|_{L^2} &\lesssim \left\| \frac{1}{r} \int_0^r f(\rho) d\rho \right\|_{L^{6,2}} = \left\| \int_0^1 f(\alpha r) d\alpha \right\|_{L^{6,2}} \\ &\lesssim \|f\|_{\dot{H}^1} \int_0^1 \alpha^{-1/2} d\alpha \lesssim \|f\|_{\dot{H}^1}. \end{aligned}$$

In the same manner one proves that for $1 \leq s < 3/2$

$$\left\| \frac{1}{r^4} \int_0^r \rho f(\rho) d\rho \right\|_{\dot{H}_{rad}^{s-2}} \lesssim \left\| \frac{1}{r} \int_0^r f(\rho) d\rho \right\|_{\dot{H}^s} \lesssim \|f\|_{\dot{H}_{rad}^s},$$

which implies that (3.8) is true for $0 \leq s < 3/2$.

The $\dot{H}^s \times \dot{H}^{s-1}$ boundedness of P_+ and P_- for $1 \leq s < 3/2$ is a consequence of their definition, of (3.8), and of Hardy's inequality $\|f/|x|\|_{\dot{H}^{s-1}} \lesssim \|f\|_{\dot{H}^s}$. The same is true for the final conclusion. \square

Next, we state the most important (and somewhat trivial) identity for outgoing solutions.

Proposition 3.8. *If u is a radial solution to the free wave equation (3.5) with outgoing initial data (u_0, u_1) , then for $r \geq t \geq 0$*

$$u(r, t) = \frac{r-t}{r} u_0(r-t)$$

and $u(r, t) \equiv 0$ for $0 \leq r \leq t$.

Note that, as t increases, an outgoing solution keeps constant sign. The computation is different, so this is not true, for negative t . Equivalently, the incoming component need not keep a constant sign for positive t .

Also note that by a direct computation one can check the outgoing property $P_-(u(t), u_t(t)) = 0$, i.e. $u_t + u_r + u/r = 0$.

Proof. This follows from the one-dimensional reduction. Indeed, let $v = T(u)$, where T is given by (3.1). Since u is outgoing, by definition v is also outgoing, i.e. $v(r, t) = v_+(r-t)$ for all $t \geq 0$ and some v_+ supported on $[0, \infty)$. Then by (3.2)

$$ru(r, t) = \int_0^r v(s, t) ds = \int_0^r v_+(s-t) ds = \int_0^{r-t} v_+(s) ds,$$

which only depends on $r-t$. Therefore $ru(r, t) = (r-t)u(r-t, 0)$. The second conclusion follows because the integral is zero when $r \leq t$. \square

This identity immediately leads to improved Strichartz and decay estimates for outgoing solutions.

Corollary 3.9 (Uniform bounds). *If u is a radial solution to the free wave equation (3.5) in three dimensions with outgoing initial data (u_0, u_1) , then if $u_0 \in L^\infty$*

$$\|u\|_{L_{t,x}^\infty} \leq \|u_0\|_{L^\infty} \tag{3.10}$$

and if $u_0 \in L^p$ then $\|u\|_{L_t^\infty L_x^p} \leq \|u_0\|_{L^p}$ for $2 \leq p \leq \infty$. In particular, the L^2 norm $\|u(x, t)\|_{L_x^2}$ is constant with respect to time for $t \geq 0$. Furthermore, $\|u\|_{L_t^\infty \dot{W}_x^{1,p}} \lesssim \|u_0\|_{\dot{W}^{1,p}}$ for $2 \leq p < 3$.

Note that the last estimate is better than one would expect from scaling.

Proof. Inequality (3.10) follows directly from Proposition 3.8, since for $0 \leq t \leq r$ $0 \leq \frac{r-t}{r} \leq 1$. Concerning the L^p norm, for $2 \leq p < \infty$

$$\begin{aligned} \|u(x, t)\|_{L_x^p}^p &= 4\pi \int_0^\infty |u(r, t)|^p r^2 dr = 4\pi \int_t^\infty \left(\frac{r-t}{r}\right)^p |u_0(r-t)|^p r^2 dr \\ &\leq 4\pi \int_0^\infty |u_0(r)|^p r^2 dr \end{aligned}$$

since $\left(\frac{r-t}{r}\right)^p r^2 \leq (r-t)^2$ for $p \geq 2$.

Note that by dominated convergence, when $2 < p < \infty$, in fact $\|u(x, t)\|_{L_x^p} \rightarrow 0$ as $t \rightarrow \infty$. When $p = \infty$ $\|u(x, t)\|_{L_x^\infty} \rightarrow 0$ as $t \rightarrow \infty$ for $u_0 \in L_0^\infty$, the closure in L^∞ of the set of bounded functions with compact support.

Regarding the Sobolev norms, again by Proposition 3.8

$$\|u(x, t)\|_{\dot{W}^{1,p}}^p = 4\pi \int_0^\infty |u_r(r, t)|^p r^2 dr = 4\pi \int_t^\infty \left| \frac{r-t}{r} (u_0)_r(r-t) + \frac{t}{r^2} u_0(r-t) \right|^p r^2 dr.$$

We bound the first term exactly as above and for the second term we use Hardy's inequality, i.e.

$$\int_t^\infty \left(\frac{t}{r^2}\right)^p |u_0(r-t)|^p r^2 dr \leq \int_0^\infty \left(\frac{|u_0(r)|}{r}\right)^p r^2 dr \lesssim \|u_0\|_{\dot{W}^{1,p}}^p$$

since $t \leq r$ and $p \geq 2$. \square

Corollary 3.10 (Decay and reversed Strichartz estimates). *Suppose that the initial data (u_0, u_1) is outgoing and $\text{supp } u_0 \subset B(0, R)$. Then for $t \geq 0$ and $2 \leq p \leq \infty$, the solution to the free wave equation (1.1) satisfies*

$$\|u(x, t)\|_{L_x^p} \lesssim \min(1, \left(\frac{R}{t}\right)^{-1+2/p}) \|u_0\|_{L^p}. \quad (3.11)$$

Suppose that $\text{supp } u_0 \subset B(0, R)$. Then for $3 < q < \infty$ (and $L^{3,\infty}$ or L^∞ at the endpoints) and $1 \leq p \leq \infty$

$$\|u\|_{L_x^{q,2} L_t^p} \lesssim R^{3/q+1/p} \|u_0\|_{L^\infty} \quad (3.12)$$

and for $1 \leq p \leq 2$

$$\|u\|_{L_x^{3,\infty} L_t^p} \lesssim R^{1/p-1/2} \|u_0\|_{L^2}.$$

More generally, suppose that $\text{supp } u_0 \subset B(0, R_1) \setminus B(0, R_2)$ for $R_1 > R_2$ and $u_0 \in L^\infty$. Then for $3 < q < \infty$ (and $L^{3,\infty}$ or L^∞ at the endpoints) and $1 \leq p \leq \infty$

$$\|u\|_{L_x^{q,2} L_t^p} \lesssim R_1^{3/q} (R_1 - R_2)^{1/p} \|u_0\|_{L^\infty}. \quad (3.13)$$

Also, for $1 \leq p \leq 2$ and $3 < q \leq \infty$ (and $L^{3,\infty}$ for $q = 3$)

$$\|u\|_{L_x^{q,2} L_t^p} \lesssim (R_1 - R_2)^{1/p-1/2} R_2^{-1+3/q} \|u_0\|_{L^2}.$$

Proof. Estimate (3.11) is an obvious consequence of Proposition 3.8 when $p = \infty$ and we interpolate with $p = 2$ (for which $\|u(x, t)\|_{L_x^2} = \|u_0\|_{L^2}$) to get all the other cases.

Next, (3.12) follows because, when $\text{supp } u_0 \subset B(0, R)$ and $u_0 \in L^\infty$, by Proposition 3.8 (where we use the fact that $\frac{r-t}{r} \leq 1$ on one hand and that $r-t \leq R$ on $\text{supp } u_0$ on the other hand)

$$\|u(r, t)\|_{L_t^p} \lesssim \min(R^{1/p} \|u_0\|_{L^\infty}, \frac{R^{1+1/p}}{r} \|u_0\|_{L^\infty}).$$

Likewise, for $1 \leq p \leq 2$

$$\|u(r, t)\|_{L_t^p} \lesssim \frac{R^{1/p-1/2}}{r} \|u_0\|_{L^2}.$$

Finally, (3.13) is true because, when $\text{supp } u_0 \subset B(0, R_1) \setminus B(0, R_2)$ and $u_0 \in L^\infty$,

$$\|u(r, t)\|_{L_t^p} \lesssim \min((R_1 - R_2)^{1/p} \|u_0\|_{L^\infty}, \frac{R_1(R_1 - R_2)^{1/p}}{r} \|u_0\|_{L^\infty}).$$

Also, for the last inequality, by Proposition 3.8

$$\|u(r, t)\|_{L_t^p} \lesssim \min\left(\frac{(R_1 - R_2)^{1/p-1/2}}{r} \|u_0\|_{L^2}, \frac{(R_1 - R_2)^{1/p-1/2}}{R_2} \|u_0\|_{L^2}\right).$$

□

We next state some Strichartz estimates that hold only for outgoing solutions. For simplicity, we state them only for the scaling-invariant norms of our problem (1.1).

Corollary 3.11 (Strichartz estimates). *For any $4 \leq N \leq \infty$, if $u_0 \in L^\infty$ and $(u_0, u_1) \in (\dot{H}^1 \times L^2)_{out}$ are radial and outgoing, then the corresponding solution u to the free wave equation (3.5) in three dimensions fulfills*

$$\|u\|_{L_t^N \dot{W}_x^{2/N, N}} + \|u\|_{L_t^{N/2} L_x^\infty} + \|u\|_{L_t^\infty \dot{W}_x^{4/N, N/2}} \lesssim \|u_0\|_{\dot{H}^1}^{4/N} \|u_0\|_{L^\infty}^{1-4/N} \quad (3.14)$$

and

$$\| |u|^N u \|_{L_t^1 \dot{H}_x^{sc-1}} \lesssim \|u_0\|_{\dot{H}^1}^{(N+1)4/N} \|u_0\|_{L_{x,t}^\infty}^{(N+1)(1-4/N)}. \quad (3.15)$$

Note that the bounds (3.14) hold for less than the full range of scaling-invariant norms.

Proof. Strichartz estimates for the free wave equation (see [GiVe] or [KeTa], as well as [KlMa] for the radial endpoint estimate) ensure that

$$\|u\|_{L_t^4 \dot{W}_x^{1/2, 4}} + \|u\|_{L_t^5 L_x^{10}} + \|u\|_{L_t^2 L_x^\infty} + \|u\|_{L_t^\infty \dot{H}_x^1} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} \lesssim \|u_0\|_{\dot{H}^1},$$

where we also used Lemma 3.7. Interpolating (see Theorems 5.1.2 and 6.4.5 in [BeLö] for the interpolation results) with the supremum estimate (3.10), we obtain that for $N \geq 4$

$$\|u\|_{L_t^N \dot{W}_x^{2/N, N}} + \|u\|_{L_t^{N/2} L_x^\infty} + \|u\|_{L_t^\infty \dot{W}_x^{4/N, N/2}} \lesssim \|u_0\|_{\dot{H}^1}^{4/N} \|u_0\|_{L^\infty}^{1-4/N},$$

which is the scaling-invariant estimate (3.14).

By the fractional Leibniz rule (only here we use that N is an integer and it is probably unnecessary), for any integer $N \geq 4$

$$\| |u|^N u \|_{L_t^1 \dot{H}_x^1} \lesssim \|u\|_{L_t^\infty \dot{H}_x^1} \|u\|_{L_t^2 L_x^\infty}^2 \|u\|_{L_{t,x}^\infty}^{N-2} \lesssim \|u_0\|_{\dot{H}^1}^3 \|u_0\|_{L^\infty}^{N-2} \quad (3.16)$$

and by Hölder's inequality

$$\| |u|^N u \|_{L_t^1 L_x^2} \lesssim \|u\|_{L_t^5 L_x^{10}}^5 \|u\|_{L_{t,x}^\infty}^{N-4} \lesssim \|u_0\|_{\dot{H}^1}^5 \|u_0\|_{L^\infty}^{N-4}. \quad (3.17)$$

In particular, since $s_c - 1 = 1/2 - 2/N$ and $3(\frac{1}{2} - \frac{2}{N}) + 5(\frac{1}{2} + \frac{2}{N}) = (N+1)4/N$, by interpolation between (3.16) and (3.17) we obtain (3.15). \square

4. STANDARD EXISTENCE RESULTS

We first state some standard Strichartz estimates, see [KeTa], that hold in scaling-invariant norms for equation (1.1).

Proposition 4.1. *Consider a solution u of the linear wave equation in three dimensions with a source term*

$$u_{tt} - \Delta u = F, \quad u(0) = u_0, \quad u_t(0) = u_1.$$

Then

$$\begin{aligned} \|u\|_{L_t^\infty \dot{H}_x^s \cap L_t^4 \dot{W}_x^{s_c-1/2,4} \cap L_t^{N/2} L_x^\infty} + \|u_t\|_{L_t^\infty \dot{H}_x^{s_c-1} \cap L_t^{4N/(N-4)} L_x^{4N/(N+4)}} &\lesssim \\ &\lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \|F\|_{L_t^1 \dot{H}_x^{s_c-1}}. \end{aligned}$$

Another simple linear estimate we shall use is

$$\left\| \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f \right\|_{L^\infty} \lesssim |t| \|f\|_{L^\infty}. \quad (4.1)$$

We next state some reversed-norm Strichartz estimates, following [BeGo]. Again we only state those estimates which hold in scaling-invariant norms for equation (1.1).

Proposition 4.2. *Consider a solution u of the linear wave equation in three dimensions with a source term*

$$u_{tt} - \Delta u = F, \quad u(0) = u_0, \quad u_t(0) = u_1.$$

Then

$$\begin{aligned} \|u\|_{L_x^{3N/2,2} L_t^\infty} &\lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \|F\|_{L_x^{\frac{3N}{2(N+1)},2} L_t^\infty}, \\ \|u\|_{L_x^\infty L_t^{N/2}} &\lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \|F\|_{L_x^{3/2,1} L_t^{N/2}}. \end{aligned} \quad (4.2)$$

Note that these reversed-norm estimates also hold (for the projection on the continuous spectrum) if the Hamiltonian is $-\Delta + V$ instead of $-\Delta$, where V is a Kato-class potential, if there are no eigenvalues or resonances in the continuous spectrum.

Remark 4.3. The following strictly stronger (in our context) inequalities are also true:

$$\|D_x^{s_c-1}u\|_{L_x^{6,2}L_t^\infty} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \|D_x^{s_c-1}F\|_{L_x^{6/5,2}L_t^\infty}$$

and

$$\|D_t^{s_c-1}u\|_{L_x^\infty L_t^2} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \|D_t^{s_c-1}F\|_{L_x^{3/2,1}L_t^2}.$$

It is also possible to base a fixed point argument on these inequalities.

Although we don't use them, we next state some standard well-posedness results for the semilinear wave equation (1.1)

$$u_{tt} - \Delta u \pm |u|^N u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1.$$

The first existence result is one that holds in the standard Strichartz norms.

Proposition 4.4. *Assume that $N > 4$ and $\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}}$ is sufficiently small (or $N = 4$ and the data are small and radially symmetric). Then equation (1.1) admits a global solution u with (u_0, u_1) as initial data, such that*

$$\begin{aligned} \|u\|_{L_t^\infty \dot{H}_x^s \cap L_t^4 \dot{W}_x^{s_c-1/2,4} \cap L_t^{N/2} L_x^\infty} + \|u_t\|_{L_t^\infty \dot{H}_x^{s_c-1} \cap L_t^{4N/(N-4)} L_x^{4N/(N+4)}} &\lesssim \\ &\lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}}. \end{aligned}$$

In addition, u scatters: there exist $(u_{0+}, u_{1+}) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ such that

$$\lim_{t \rightarrow \infty} \|(u(t), u_t(t)) - \Phi(t)(u_{0+}, u_{1+})\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} = 0$$

and likewise as $t \rightarrow -\infty$.

More generally, if $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ are not small, then there exist an interval $I = (-T, T)$ with $T > 0$ and a solution u to (1.1) defined on $\mathbb{R}^3 \times I$, having (u_0, u_1) as initial data, such that

$$\begin{aligned} &\|u\|_{L_t^\infty \dot{H}_x^s(\mathbb{R}^3 \times I) \cap L_t^4 \dot{W}_x^{s_c-1/2,4}(\mathbb{R}^3 \times I) \cap L_t^{N/2} L_x^\infty(\mathbb{R}^3 \times I)} + \\ &+ \|u_t\|_{L_t^\infty \dot{H}_x^{s_c-1}(\mathbb{R}^3 \times I) \cap L_t^{4N/(N-4)} L_x^{4N/(N+4)}(\mathbb{R}^3 \times I)} < \infty. \end{aligned}$$

Proof. This is a consequence of the standard Strichartz estimates for the free wave equation of [KeTa], see Proposition 4.1.

The proof works by a contraction argument in the $L_t^\infty \dot{H}_x^s \cap L_t^4 \dot{W}_x^{s_c-1/2,4} \cap L_t^{N/2} L_x^\infty$ norm. Indeed, note that the nonlinearity can be bounded in the dual Strichartz norm by

$$\||u|^N u\|_{L_t^1 \dot{H}_x^{s_c-1}} \lesssim \|u\|_{L_t^{N/2} L_x^\infty}^{N/2} \|u\|_{L_t^\infty L_x^{3N/2}}^{N/2} \|u\|_{L_t^\infty \dot{H}_x^s} \lesssim \|u\|_{L_t^{N/2} L_x^\infty}^{N/2} \|u\|_{L_t^\infty \dot{H}_x^s}^{N/2+1}.$$

Concerning scattering, we define

$$\begin{aligned} u_{0+} &:= u_0 - \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} (|u(s)|^N u(s)) ds, \\ u_{1+} &:= u_1 + \int_0^\infty \cos(s\sqrt{-\Delta}) (|u(s)|^N u(s)) ds. \end{aligned}$$

Then

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \left(\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \cos(s\sqrt{-\Delta}) - \cos(t\sqrt{-\Delta}) \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} \right) (|u(s)|^N u(s)) ds,$$

so

$$u(t) - \Phi_0(t)(u_{0+}, u_{1+}) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \int_t^\infty \cos(s\sqrt{-\Delta}) (|u(s)|^N u(s)) ds - \cos(t\sqrt{-\Delta}) \int_t^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} (|u(s)|^N u(s)) ds,$$

where $\Phi_0(t)$ is the first component of $\Phi(t)$. This expression goes to zero in the \dot{H}^{s_c} norm. The same is true for u_t .

In case the initial data is not small, the global $L_t^4 \dot{W}_x^{s_c-1/2,4} \cap L_t^{N/2} L_x^\infty$ Strichartz norm of its linear development is still finite, hence it becomes small on some sufficiently small interval $(-T, T)$, and we run the contraction argument on that interval. In the same way one can prove the uniqueness of the solution in $L_t^\infty \dot{H}_x^s(\mathbb{R}^3 \times I) \cap L_t^{N/2} L_x^\infty(\mathbb{R}^3 \times I)$, where I is the maximal interval on which the solution is defined. \square

The second existence result holds in the reversed Strichartz norms introduced in [BeGo], being a straightforward generalization of Proposition 5 from that paper.

Proposition 4.5. *Assume that $N \geq 4$ and $\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}}$ is sufficiently small. Then equation (1.1) admits a global solution u with (u_0, u_1) as initial data, such that*

$$\|u\|_{L_x^{3N/2} L_t^\infty \cap L_x^\infty L_t^{N/2}} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}}.$$

Moreover, if the initial data $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ are not small, there exist an interval $I = (-T, T)$ and a solution u to (1.1) on $\mathbb{R}^3 \times I$ such that

$$\|u\|_{L_x^{3N/2} L_t^\infty(\mathbb{R}^3 \times I) \cap L_x^\infty L_t^{N/2}(\mathbb{R}^3 \times I)} < \infty.$$

Proof. The proof is based on the reversed-norm Strichartz estimates of Proposition 4.2 and on a contraction argument in the $L_x^{3N/2} L_t^\infty \cap L_x^\infty L_t^{N/2}$ norm. Indeed, note that the nonlinearity can be bounded in the dual Strichartz norm by

$$\begin{aligned} \| |u|^N u \|_{L_x^{\frac{3N}{2(N+1)},2} L_t^\infty} &\lesssim \|u\|_{L_x^{3N/2,2} L_t^\infty}^{N+1}, \\ \| |u|^N u \|_{L_x^{3/2,1} L_t^{N/2}} &\lesssim \|u\|_{L_x^{3N/2,2} L_t^\infty}^N \|u\|_{L_x^\infty L_t^{N/2}}. \end{aligned}$$

One can obtain the large data local well-posedness result as follows: by means of smooth cutoffs, we restrict the initial data to sets on which their norm is small and solve the initial value problem with this restricted data.

The solutions obtained will agree on some small time interval with the solution of the original problem due to the finite speed of propagation. Since there is a lower bound on how small the diameter of the sets is required to be, by piecing together all these partial solutions we obtain a global in space solution on some nonempty time interval.

In the same way one can prove that the solution is unique in $L_x^{3N/2} L_t^\infty(\mathbb{R}^3 \times I)$, where I is a small interval (or $I = \mathbb{R}$ for small norm solutions). \square

5. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. This is a direct consequence of the standard existence theory, in view of the Strichartz estimates (3.14) and (3.15).

Explicitly, we write the solution u as the sum of the free evolution of the outgoing initial data and a small perturbation, to which we apply a contraction argument. Let $u(x, t) = v(x, t) + w(x, t)$, where

$$v_{tt} - \Delta v = 0, \quad v(0) = v_0, \quad v_t(0) = v_1$$

and

$$w_{tt} - \Delta w \pm |v + w|^N(v + w) = 0, \quad w(0) = w_0, \quad w_t(0) = w_1. \quad (5.1)$$

We linearize equation (5.1) by writing it as

$$w_{tt} - \Delta w \pm |v + \tilde{w}|^N(v + \tilde{w}) = 0, \quad w(0) = w_0, \quad w_t(0) = w_1, \quad (5.2)$$

and solving for $w = F(\tilde{w})$, while treating \tilde{w} as given. The subsequent argument is the same if instead of null initial data we take small $(w_0, w_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ initial data in (5.1), i.e. if we allow for a small perturbation of the outgoing initial data (v_0, v_1) .

By a standard contraction argument we proceed to show that there exists \tilde{w} of bounded $L_t^\infty \dot{H}_x^{s_c} \cap L_t^4 \dot{W}_x^{s_c-1/2, 4} \cap L_t^{N/2} L_x^\infty$ norm such that $\tilde{w} = F(\tilde{w})$.

The source term in equation (5.1) is $|v|^N v$, which is controlled in the appropriate norm by (3.15): if we denote

$$K := \|v_0\|_{\dot{H}^1}^{4/N} \|v_0\|_{L_{t,x}^\infty}^{1-4/N},$$

then

$$\| |v|^N v \|_{L_t^1 \dot{H}_x^{s_c-1}} \lesssim K^{N+1}. \quad (5.3)$$

The other (mixed) terms are bounded in the critical norm by (3.14) because for $N \leq 12$

$$s_c - 1 = 1/2 - 2/N \leq 4/N. \quad (5.4)$$

Indeed,

$$\begin{aligned} |v + \tilde{w}|^N(v + \tilde{w}) - |v|^N v &= (|v + \tilde{w}|^N - |v|^N)(v + \tilde{w}) + |v|^N \tilde{w} \\ &= \tilde{w} \left(\int_0^1 N |v + \alpha \tilde{w}|^{N-2} (v + \alpha \tilde{w}) d\alpha \right) (v + \tilde{w}) + |v|^N \tilde{w}. \end{aligned}$$

Furthermore, note that

$$\|u^{N+1}\|_{L_t^1 \dot{H}_x^{s_c-1}} \lesssim \|u\|_{L_t^{N/2} L_x^\infty}^{N/2} \|u\|_{L_t^{3N/2} L_x^\infty}^{N/2} \|u\|_{L_t^\infty \dot{W}^{s_c-1,6}} \lesssim \|u\|_{L_t^{N/2} L_x^\infty}^{N/2} \|u\|_{L_t^\infty \dot{W}^{s_c-1,6}}^{N/2+1}$$

and more generally (here we use that N is an integer, though it is probably unnecessary)

$$\|u_1 \dots u_{N+1}\|_{L_t^1 \dot{H}_x^{s_c-1}} \lesssim \|u_1\|_{L_t^\infty \dot{W}^{s_c-1,6} \cap L_t^{N/2} L_x^\infty} \dots \|u_{N+1}\|_{L_t^\infty \dot{W}^{s_c-1,6} \cap L_t^{N/2} L_x^\infty}.$$

Then for $0 \leq \alpha \leq 1$

$$\begin{aligned} & \|\tilde{w}|v + \alpha \tilde{w}|^{N-2}(v + \alpha \tilde{w})(v + \tilde{w})\|_{L_t^1 \dot{H}_x^{s_c-1}} \lesssim \\ & \lesssim \|\tilde{w}\|_{L_t^\infty \dot{W}^{s_c-1,6} \cap L_t^{N/2} L_x^\infty} (\|v\|_{L_t^\infty \dot{W}^{s_c-1,6} \cap L_t^{N/2} L_x^\infty}^N + \|\tilde{w}\|_{L_t^\infty \dot{W}^{s_c-1,6} \cap L_t^{N/2} L_x^\infty}^N) \\ & \lesssim \|\tilde{w}\|_{L_t^\infty \dot{H}^{s_c} \cap L_t^{N/2} L_x^\infty} (\|v\|_{L_t^\infty \dot{W}^{4/N, N/2} \cap L_t^{N/2} L_x^\infty}^N + \|\tilde{w}\|_{L_t^\infty \dot{H}^{s_c} \cap L_t^{N/2} L_x^\infty}^N), \end{aligned} \quad (5.5)$$

where $\dot{H}^{s_c} \subset \dot{W}^{s_c-1,6}$ and $\dot{W}^{4/N, N/2} \subset \dot{W}^{s_c-1,6}$. A similar estimate holds for $|v|^N w$.

Note that, for (5.5) to hold, each factor on the left-hand side must have at least $s_c - 1$ derivatives. There are some monomials in (5.5) with only one power of v ; since all the other factors (powers of w) can only be bounded in scaling-invariant norms, due to scaling we must also bound v in a scaling-invariant norm. Since by (3.14) v only has $\frac{4}{N}$ derivatives in a scaling-invariant norm, condition (5.4) is necessary.

From (5.3) and (5.5), combined with standard Strichartz estimates, we get that

$$\|w\|_{L_t^\infty \dot{H}_x^{s_c} \cap L_t^4 \dot{W}_x^{s_c-1/2,4} \cap L_t^{N/2} L_x^\infty} \lesssim \|w_0\|_{\dot{H}^{s_c}} + \|w_1\|_{\dot{H}^{s_c-1}} + K^{N+1} + \|\tilde{w}\|_{L_t^\infty \dot{H}_x^{s_c} \cap L_t^{N/2} L_x^\infty}^{N+1}.$$

If we assume that $\|\tilde{w}\|_{L_t^\infty \dot{H}_x^{s_c} \cap L_t^{N/2} L_x^\infty} \leq K$ and that w_0 , w_1 , and K are sufficiently small, it follows that $\|w\|_{L_t^\infty \dot{H}_x^{s_c} \cap L_t^{N/2} L_x^\infty} \leq K$ as well.

Note that

$$\begin{aligned} & |v + \tilde{w}_1|^N (v + \tilde{w}_1) - |v + \tilde{w}_2|^N (v + \tilde{w}_2) = \\ & = \int_0^1 \frac{d}{d\alpha} (|v + \alpha \tilde{w}_1 + (1 - \alpha) \tilde{w}_2|^N (v + \alpha \tilde{w}_1 + (1 - \alpha) \tilde{w}_2)) d\alpha \\ & = (\tilde{w}_1 - \tilde{w}_2) \int_0^1 (N + 1) |v + \alpha \tilde{w}_1 + (1 - \alpha) \tilde{w}_2|^{N-1} d\alpha. \end{aligned}$$

One then shows that, given two pairs w^1, \tilde{w}^1 and w^2, \tilde{w}^2 that both fulfill (5.2),

$$\begin{aligned} & \|w^1 - w^2\|_{L_t^\infty \dot{H}_x^{s_c} \cap L_t^4 \dot{W}_x^{s_c-1/2,4} \cap L_t^{N/2} L_x^\infty} \lesssim \\ & \lesssim \|\tilde{w}^1 - \tilde{w}^2\|_{L_t^\infty \dot{H}_x^{s_c} \cap L_t^{N/2} L_x^\infty} (K^N + \|\tilde{w}^1\|_{L_t^\infty \dot{H}_x^{s_c} \cap L_t^{N/2} L_x^\infty}^N + \|\tilde{w}^2\|_{L_t^\infty \dot{H}_x^{s_c} \cap L_t^{N/2} L_x^\infty}^N). \end{aligned}$$

It follows that the mapping $\tilde{w} \mapsto w$ is a contraction in the sphere of radius K in $L_t^\infty \dot{H}_x^{s_c} \cap L_t^{N/2} L_x^\infty$ when w_0, w_1 , and K are sufficiently small. The fixed

point $w = \tilde{w}$ then gives rise to a global solution $u = v + w$ to (1.1). As a byproduct we can also obtain the $L_t^\infty \dot{H}^{s_c-1}$ norm of w_t .

Estimate (1.3) is true because we can separately bound v (by (3.14)) and w (by the fixed point argument) in the $L_t^{N/2} L_x^\infty$ norm.

Concerning scattering, define

$$\begin{aligned} w_{0+} &:= w_0 - \int_0^\infty \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} (|u(s)|^N u(s)) ds, \\ w_{1+} &:= w_1 + \int_0^\infty \cos(s\sqrt{-\Delta}) (|u(s)|^N u(s)) ds. \end{aligned}$$

Since $|u(s)|^N u(s) \in L_t^1 \dot{H}_x^{s_c-1}$, it is easy to show (1.4).

If the initial data are in $((\dot{H}^1 \cap L^\infty) \times L^2)_{out} + \dot{H}^s \times \dot{H}^{s-1}$, but not small, then still the norm $\|v\|_{L_t^{N/2} L_x^\infty} < \infty$ is finite, so there exists an interval $I = [0, T]$ on which $\|v\|_{L_t^{N/2} L_x^\infty(\mathbb{R}^3 \times I)}$ is small and same for the linear evolution of (w_0, w_1) . We then run the previous argument on this interval.

In the same manner one can prove the uniqueness of the solution in $L_t^\infty \dot{H}_x^{s_c}(\mathbb{R}^3 \times I) \cap L_t^{N/2} L_x^\infty(\mathbb{R}^3 \times I)$, where I is the maximal interval of existence. \square

Proof of Corollary 1.2. This follows from Theorem 1.1 and the \mathbb{R}^3 radial \dot{H}^1 Sobolev embedding

$$|u_0(r)| \lesssim r^{-1/2} \|u_0\|_{\dot{H}_{rad}^1}.$$

Given that u_0 is supported outside the sphere $B(0, R)$, this embedding implies that

$$\|u_0\|_{L^\infty} \lesssim R^{-1/2} \|u_0\|_{\dot{H}_{rad}^1}.$$

Therefore $\|u_0\|_{\dot{H}^1}^{4/N} \|u_0\|_{L^\infty}^{1-4/N} \lesssim \|u_0\|_{\dot{H}^1} R^{-(1-4/N)/2}$. The conclusion follows by applying Theorem 1.1. \square

For the sake of completeness, we also state some local existence results. We begin with a simple, but weak result that holds for bounded initial data.

Proposition 5.1. *Suppose that $N > 0$, the initial data (u_0, u_1) are radial and outgoing, and $u_0 \in L^\infty$. Then there exists a corresponding solution u to (1.1) on $\mathbb{R}^3 \times I$, $I = [0, T]$, such that $T \geq C \|u_0\|^{-N/2}$ and*

$$\|u\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)} \lesssim \|u_0\|_{L^\infty}.$$

Note that one cannot repeat this argument for later initial times because the nonlinearity generates incoming terms and for incoming initial data it is not enough for it to be in L^∞ .

Proof of Proposition 5.1. We apply a fixed point argument. Linearize equation (1.1) to

$$u_{tt} - \Delta u \pm |\tilde{u}|^N \tilde{u} = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (5.6)$$

Then, taking into account (3.10) and (4.1),

$$\|u\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)} \lesssim \|u_0\|_{L^\infty} + T^2 \|\tilde{u}\|^N \|\tilde{u}\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)} \lesssim \|u_0\|_{L^\infty} + T^2 \|\tilde{u}\|_{L_{t,x}^\infty}^{N+1}.$$

Thus, if $\|\tilde{u}\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)} \lesssim \|u_0\|_{L^\infty}$ and $T \leq c \|u_0\|_{L_{t,x}^\infty}^{-N/2}$ with c sufficiently small, we retrieve the same conclusion for u . In addition, the mapping $\tilde{u} \mapsto u$ is a contraction. Indeed, given two pairs \tilde{u}^1 and u^1 , respectively \tilde{u}^2 and u^2 , which fulfill (5.6),

$$\|u^1 - u^2\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)} \lesssim T^2 \|\tilde{u}^1 - \tilde{u}^2\|_{L_{t,x}^\infty} (\|\tilde{u}^1\|_{L_{t,x}^\infty}^N + \|\tilde{u}^2\|_{L_{t,x}^\infty}^N).$$

Thus, if $T \leq c \|u_0\|_{L_{t,x}^\infty}^{-N/2}$ with sufficiently small c , then the mapping $\tilde{u} \mapsto u$ is a contraction on $\{u \mid \|u\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)} \lesssim \|u_0\|_{L_{t,x}^\infty}\}$. The fixed point is a solution of (1.1) with the desired properties. \square

We next state another local existence result in the subcritical sense, for large $((\dot{H}^1 \cap L^\infty) \times L^2)_{out} + (\dot{H}^2 \cap \dot{H}^1) \times (\dot{H}^1 \cap L^2)$ initial data (and in particular for $(\dot{H}^1 \cap L^\infty) \times L^2_{out}$ initial data). The solution remains in the same class for some finite positive time.

Proposition 5.2. *Assume $N \geq 2$ and consider initial data $(u_0, u_1) = (v_0, v_1) + (w_0, w_1)$, where $(v_0, v_1) \in ((\dot{H}^1 \cap L^\infty) \times L^2)_{out}$ are radial and outgoing and $(w_0, w_1) \in (\dot{H}^2 \cap \dot{H}^1) \times (\dot{H}^1 \cap L^2)$ are radial. Then there exists a corresponding solution u to (1.1) on $\mathbb{R}^3 \times I$, where $I = [0, T]$ and*

$$T \geq C(\|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + \|v_0\|_{\dot{H}^1 \cap L^\infty})^{-N},$$

such that $u = v + w$ and

$$\|(v, v_t)\|_{L_t^\infty((\dot{H}^1 \cap L^\infty) \times L^2)_{out}(\mathbb{R}^3 \times I)} \lesssim \|v_0\|_{\dot{H}^1 \cap L^\infty}, \quad (5.7)$$

$$\begin{aligned} \|(w, w_t)\|_{L_t^\infty(\dot{H}_x^2 \cap \dot{H}_x^1 \times \dot{H}_x^1 \cap L_x^2)(\mathbb{R}^3 \times I)} &\lesssim \\ &\lesssim \|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + \|v_0\|_{\dot{H}^1 \cap L^\infty}. \end{aligned}$$

Assume in addition that $N \geq 4$ and

$$\begin{aligned} &\|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + \|v_0\|_{\dot{H}^1}^{5N-4} \|v_0\|_{L^\infty}^{(N-4)(N-1)} + \\ &+ \|v_0\|_{\dot{H}^1}^2 \|v_0\|_{L^\infty}^{N-2} + (\|w_0\|_{\dot{H}^2 \cap \dot{H}^1}^{N-1} + \|w_1\|_{\dot{H}^1 \cap L^2}^{N-1}) \|v_0\|_{\dot{H}^1} << 1. \end{aligned} \quad (5.8)$$

Then there exists a global solution u , forward in time, with this initial data, such that $u = v + w$, v fulfills (5.7), and

$$\begin{aligned} \|(w, w_t)\|_{L_t^\infty(\dot{H}_x^2 \cap \dot{H}_x^1 \times \dot{H}_x^1 \cap L_x^2)} + \|w\|_{L_t^2 L_x^\infty} &\lesssim \|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + \\ &+ \|v_0\|_{\dot{H}^1}^5 \|v_0\|_{L^\infty}^{N-4} + \|v_0\|_{\dot{H}^1}^3 \|v_0\|_{L^\infty}^{N-2}. \end{aligned}$$

Proof of Proposition 5.2. As before, we write the solution as a sum of two terms, $u(x, t) = v(x, t) + w(x, t)$, where v is the linear evolution of (v_0, v_1) and w is the contribution of (w_0, w_1) and of the nonlinear terms:

$$v_{tt} - \Delta v = 0, \quad v(0) = v_0, \quad v_t(0) = v_1$$

and we linearize the second equation to (5.2), that is

$$w_{tt} - \Delta w \pm |v + \tilde{w}|^N(v + \tilde{w}) = 0, \quad w(0) = w_0, \quad w_t(0) = w_1.$$

Then clearly (v, v_t) satisfy (5.7), see (3.10) and Lemma 3.7. In addition,

$$\begin{aligned} \|(w, w_t)\|_{L_t^\infty(\dot{H}_x^2 \cap \dot{H}_x^1 \times \dot{H}_x^1 \cap L_x^2)(\mathbb{R}^3 \times I)} &\lesssim \\ &\lesssim \|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + T \| |v + \tilde{w}|^N(v + \tilde{w}) \|_{L_t^\infty(\dot{H}_x^1 \cap L_x^2)(\mathbb{R}^3 \times I)}. \end{aligned}$$

Here

$$\begin{aligned} \| |v + \tilde{w}|^N(v + \tilde{w}) \|_{L_t^\infty L_x^2(\mathbb{R}^3 \times I)} &\lesssim \|v\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)}^{N-2} \|v\|_{L_t^\infty \dot{H}_x^1(\mathbb{R}^3 \times I)}^3 \\ &\quad + \|\tilde{w}\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)}^{N-2} \|\tilde{w}\|_{L_t^\infty \dot{H}_x^1(\mathbb{R}^3 \times I)}^3 \end{aligned}$$

and

$$\begin{aligned} \| |v + \tilde{w}|^N(v + \tilde{w}) \|_{L_t^\infty \dot{H}_x^1(\mathbb{R}^3 \times I)} &\lesssim (\|v\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)}^N + \|\tilde{w}\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)}^N) \\ &\quad (\|v\|_{L_t^\infty \dot{H}_x^1(\mathbb{R}^3 \times I)} + \|\tilde{w}\|_{L_t^\infty \dot{H}_x^1(\mathbb{R}^3 \times I)}). \end{aligned}$$

Also note that

$$\|\tilde{w}\|_{L_{t,x}^\infty(\mathbb{R}^3 \times I)} \lesssim \|\tilde{w}\|_{L_t^\infty(\dot{H}_x^2 \cap \dot{H}_x^1)(\mathbb{R}^3 \times I)}. \quad (5.9)$$

In conclusion, if

$$\|\tilde{w}\|_{L_t^\infty(\dot{H}_x^2 \cap \dot{H}_x^1)(\mathbb{R}^3 \times I)} \lesssim \|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + \|v_0\|_{\dot{H}^1 \cap L^\infty}$$

and if

$$T \leq c(\|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + \|v_0\|_{\dot{H}^1 \cap L^\infty})^{-N}$$

with c sufficiently small, then we retrieve the same conclusion for w . Under the same condition one can prove that the mapping $\tilde{w} \mapsto w$ is a contraction on the set $\{w \mid \|\tilde{w}\|_{L_t^\infty(\dot{H}_x^2 \cap \dot{H}_x^1)(\mathbb{R}^3 \times I)} \lesssim \|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + \|v_0\|_{\dot{H}^1 \cap L^\infty}\}$. The fixed point w gives rise to a solution $u = v + w$ with the required properties. In particular, we also retrieve a bound for w_t .

For the global existence result, we use the following estimates:

$$\|(w, w_t)\|_{L_t^\infty(\dot{H}_x^1 \times L_x^2)} + \|w\|_{L_t^2 L_x^\infty} \lesssim \|w_0\|_{\dot{H}^1} + \|w_1\|_{L^2} + \| |v + \tilde{w}|^N(v + \tilde{w}) \|_{L_t^1 L_x^2},$$

where

$$\| |v + \tilde{w}|^N(v + \tilde{w}) \|_{L_t^1 L_x^2} \lesssim \|v\|_{L_t^\infty \dot{H}_x^1}^3 \|v\|_{L_t^2 L_x^\infty}^2 \|v\|_{L_{t,x}^\infty}^{N-4} + \|\tilde{w}\|_{L_t^\infty \dot{H}_x^1}^3 \|\tilde{w}\|_{L_t^2 L_x^\infty}^2 \|\tilde{w}\|_{L_{t,x}^\infty}^{N-4},$$

together with

$$\|(w, w_t)\|_{L_t^\infty(\dot{H}^2 \times \dot{H}^1)} \lesssim \|w_0\|_{\dot{H}^2} + \|w_1\|_{\dot{H}^1} + \| |v + \tilde{w}|^N(v + \tilde{w}) \|_{L_t^1 \dot{H}_x^1},$$

where

$$\| |v + \tilde{w}|^N(v + \tilde{w}) \|_{L_t^1 \dot{H}_x^1} \lesssim (\|v\|_{L_t^2 L_x^\infty}^2 \|v\|_{L_{t,x}^\infty}^{N-2} + \|\tilde{w}\|_{L_t^2 L_x^\infty}^2 \|\tilde{w}\|_{L_{t,x}^\infty}^{N-2}) (\|v\|_{L_t^\infty \dot{H}_x^1} + \|\tilde{w}\|_{L_t^\infty \dot{H}_x^1}).$$

Also note (5.9) and that $\|v\|_{L_{t,x}^\infty} \leq \|v_0\|_{L^\infty}$ and $\|v\|_{L_t^\infty \dot{H}_x^1} \lesssim \|v_0\|_{\dot{H}^1}$. It follows that whenever

$$\|\tilde{w}\|_{L_t^\infty(\dot{H}_x^2 \cap \dot{H}_x^1) \cap L_t^2 L_x^\infty} \leq \epsilon \ll 1 \quad (5.10)$$

and

$$\begin{aligned} & \|w_0\|_{\dot{H}^2 \cap \dot{H}^1} + \|w_1\|_{\dot{H}^1 \cap L^2} + \|v_0\|_{\dot{H}^1}^5 \|v_0\|_{L^\infty}^{N-4} + \|v_0\|_{\dot{H}^1}^3 \|v_0\|_{L^\infty}^{N-2} \lesssim c\epsilon, \\ & \|v_0\|_{\dot{H}^1}^2 \|v_0\|_{L^\infty}^{N-2} + \epsilon^{N-1} \|v_0\|_{\dot{H}^1} << 1, \end{aligned}$$

with c sufficiently small (not depending on ϵ), then we retrieve the same conclusion (5.10) for w .

In particular, for this to happen it is necessary that $(\|v_0\|_{\dot{H}^1}^5 \|v_0\|_{L^\infty}^{N-4})^{N-1} \|v_0\|_{\dot{H}^1} << 1$, which is part of our condition (5.8).

Next, we prove that the mapping $\tilde{w} \mapsto w$ is a contraction. In the same manner as above it can be shown that, when w^1 and \tilde{w}^1 , respectively w^2 and \tilde{w}^2 , satisfy the linearized equation (5.2) and condition (5.10), then

$$\|w^1 - w^2\|_{L_t^\infty \dot{H}_x^1 \cap L_t^2 L_x^\infty} \lesssim \|\tilde{w}^1 - \tilde{w}^2\|_{L_t^\infty \dot{H}_x^1} (\|v_0\|_{\dot{H}^1}^4 \|v_0\|_{L^\infty}^{N-4} + \epsilon^N)$$

and

$$\|w^1 - w^2\|_{L_t^\infty \dot{H}_x^2} \lesssim \|\tilde{w}^1 - \tilde{w}^2\|_{L_t^\infty (\dot{H}_x^2 \cap \dot{H}_x^1)} (\|v_0\|_{\dot{H}^1}^2 \|v_0\|_{L^\infty}^{N-2} + \|v_0\|_{\dot{H}^1}^3 \|v_0\|_{L^\infty}^{N-3} + \epsilon^N).$$

Thus, as long as ϵ is sufficiently small and

$$\|v_0\|_{\dot{H}^1}^2 \|v_0\|_{L^\infty}^{N-2} + \|v_0\|_{\dot{H}^1}^3 \|v_0\|_{L^\infty}^{N-3} << 1,$$

the mapping is a contraction. The ensuing fixed point w gives rise to a solution u of (1.1) with the desired properties.

As part of the contraction argument we can also bound w_t . Putting together all the conditions we use, we obtain (5.8). \square

We continue with the proof of Theorem 1.4, concerning global existence for bounded compact support initial data.

Proof of Theorem 1.4. This follows by a standard fixed point argument in the $L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}$ norm.

Let $u(x, t) = v(x, t) + w(x, t)$, where

$$v_{tt} - \Delta v = 0, \quad v(0) = u_0, \quad v_t(0) = u_1$$

and

$$w_{tt} - \Delta w + |v + w|^N (v + w) = 0, \quad w(0) = 0, \quad w_t(0) = 0. \quad (5.11)$$

As in the proof of Theorem 1.1, we linearize (5.11) to

$$w_{tt} - \Delta w \pm |v + \tilde{w}|^N (v + \tilde{w}) = 0, \quad w(0) = 0, \quad w_t(0) = 0$$

and then we prove by a contraction argument that there exists $\tilde{w} \in L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}$ for which $w = \tilde{w}$.

Note that by (3.12)

$$\|v\|_{L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}} \lesssim R^{2/N} \|u_0\|_{L^\infty} := K.$$

Then, since the initial data are zero,

$$\begin{aligned} \|w\|_{L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}} &\lesssim \| |v + \tilde{w}|^N (v + \tilde{w}) \|_{L_x^{\frac{3N}{2(N+1)},2} L_t^\infty \cap L_x^{3/2,1} L_t^{N/2}} \\ &\lesssim K^{N+1} + \|\tilde{w}\|_{L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}}^{N+1}. \end{aligned}$$

Thus, when K is small, the mapping $\tilde{w} \mapsto w$ leaves a sufficiently small sphere in $L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}$ invariant.

Furthermore, considering two auxiliary functions \tilde{w}_1 and \tilde{w}_2 that give rise to solutions w_1 , respectively w_2 ,

$$\begin{aligned} \|w_1 - w_2\|_{L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}} &\lesssim \\ &\lesssim \| |v + \tilde{w}_1|^N (v + \tilde{w}_1) - |v + \tilde{w}_2|^N (v + \tilde{w}_2) \|_{L_x^{\frac{3N}{2(N+1)},2} L_t^\infty \cap L_x^{3/2,1} L_t^{N/2}} \\ &\lesssim \|\tilde{w}_1 - \tilde{w}_2\|_{L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}} (K^N + \|\tilde{w}_1\|_{L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}}^N + \|\tilde{w}_2\|_{L_x^{3N/2,2} L_t^\infty \cap L_x^\infty L_t^{N/2}}^N). \end{aligned}$$

Thus the mapping $\tilde{w} \mapsto w$ is a contraction in a sufficiently small sphere when K is also small. It therefore has a fixed point $w = \tilde{w}$, such that $u = v + w$ is a solution to (1.1). \square

We finally construct true large initial data global solutions to (1.1).

Proof of Theorem 1.5. For $\alpha > 0$ and $\epsilon \ll 1$, consider outgoing initial data (u_0, u_1) supported on $\overline{B(0, 1 + \epsilon)} \setminus B(0, 1)$, such that $u_0(r) = L\epsilon^{-\alpha}$ for $r \in [1, 1 + \epsilon]$. Then $\|u_0\|_{L^\infty} \sim L\epsilon^{-\alpha}$ and $\|u_0\|_{L^{1/\alpha}} \sim L$. Let v be the linear evolution of (u_0, u_1) , that is

$$v_{tt} - \Delta v = 0, \quad v(0) = u_0, \quad v_t(0) = u_1.$$

By (3.13)

$$\|v\|_{L_x^{3N/2,2} L_t^\infty} \lesssim \|u_0\|_{L^\infty} \sim L\epsilon^{-\alpha}, \quad \|v\|_{L_x^\infty L_t^{N/2}} \lesssim \epsilon^{2/N} \|u_0\|_{L^\infty} \sim L\epsilon^{2/N-\alpha}.$$

Letting ϵ go to zero, we cannot make the scaling-invariant $L_x^{3N/2,2} L_t^\infty$ reversed Strichartz norm of v small. This is why we examine one more iterate in the nonlinear contraction scheme.

Note that by Proposition 3.8 $v(r, t)$ is supported on $\overline{B(1 + t + \epsilon)} \setminus B(1 + t)$ and

$$v(1 + t + a, t) \lesssim \frac{L\epsilon^{-\alpha}}{1 + t + a} \leq \frac{L\epsilon^{-\alpha}}{1 + t}.$$

Let $n > 2$. Therefore, $v^n(r = 1 + t + a, t)$ is supported on $\overline{B(1 + t + \epsilon)} \setminus B(1 + t)$ and bounded by $\frac{L^n \epsilon^{-n\alpha}}{(1 + t + a)^n}$. To help with computations, we write this bound as

$$v^n(1 + t + a, t) \lesssim \int_0^\epsilon \frac{L^n \epsilon^{-n\alpha}}{(1 + t + a)^{n-1}} \frac{\sin((1 + t + a)\sqrt{-\Delta})}{\sqrt{-\Delta}} \delta_0 da,$$

where δ_0 is Dirac's delta and we have taken advantage of the special form of the kernel of $\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(x, y) = \frac{1}{4\pi t} \delta_{|x-y|=t}$ for $t \geq 0$.

Let us estimate the Duhamel term

$$\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} v^n(s) ds. \quad (5.12)$$

Since we use absolute values, not cancellations, we bound this from above by

$$\begin{aligned} & \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} \frac{L^n \epsilon^{-n\alpha}}{(1+s)^{n-1}} \int_0^\epsilon \frac{\sin((1+s+a)\sqrt{-\Delta})}{\sqrt{-\Delta}} \delta_0 da ds = \\ & = \int_0^\epsilon \int_0^t \frac{L^n \epsilon^{-n\alpha}}{(1+s)^{n-1}} \frac{1}{2} \left(\frac{\cos((t-1-2s-a)\sqrt{-\Delta})}{-\Delta} - \frac{\cos((1+t+a)\sqrt{-\Delta})}{-\Delta} \right) \delta_0 ds da. \end{aligned}$$

Note that $\frac{\cos(t\sqrt{-\Delta})}{-\Delta}(x, y) = \frac{1}{4\pi|x-y|} \chi_{|x-y| \geq t}$. We obtain a bound of

$$\begin{aligned} & \int_0^\epsilon \int_0^t \frac{L^n \epsilon^{-n\alpha}}{(1+s)^{n-1}} \frac{1}{r} \chi_{[|t-1-2s-a|, 1+t+a]}(r) ds da = \\ & = \int_0^\epsilon \int_{\max(0, \frac{t-1-a-r}{2})}^{\min(t, \frac{t-1-a+r}{2})} \frac{L^n \epsilon^{-n\alpha}}{(1+s)^{n-1}} ds \frac{\chi_{[\max(0, 1+a-t), 1+t+a]}(r)}{r} da \\ & \lesssim \int_0^\epsilon \frac{\chi_{[\max(0, 1+a-t), 1+t+a]}(r)}{r} \left(\frac{L^n \epsilon^{-n\alpha}}{(1 + \max(0, \frac{t-1-a-r}{2}))^{n-2}} - \frac{L^n \epsilon^{-n\alpha}}{(1 + \frac{t-1-a+r}{2})^{n-2}} \right) da \\ & \lesssim L^n \epsilon^{1-n\alpha} \chi_{[0, 2+t]}(r) \min\left(\frac{1}{r}, 1\right), \end{aligned} \quad (5.13)$$

the second part using the mean value theorem.

It follows that

$$\|(5.12)\|_{\langle x \rangle^{-1} L_{t,x}^\infty} \lesssim L^n \epsilon^{1-n\alpha}.$$

Setting $n = N + 1$, we obtain for example that this norm can be made arbitrarily small by letting ϵ go to zero if $\alpha < \frac{1}{N+1}$.

We write the solution u as a sum of two parts, $u(x, t) = v(x, t) + w(x, t)$, where v is the linear evolution of the initial data and w is the contribution of the nonlinear terms:

$$w_{tt} - \Delta w \pm (v + w)^{N+1} = 0, \quad w(0) = 0, \quad w_t(0) = 0. \quad (5.14)$$

Recall that for simplicity we assumed that N is even.

We then have to obtain similar bounds for the terms

$$\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (v^n(s) w^{N+1-n}(s)) ds \quad (5.15)$$

for $0 \leq n \leq N + 1$. For $n > 0$ we proceed in the same manner as in (5.13). Note that $v^n(t) w^{N+1-n}(t)$ is supported on $\overline{B(1+t+\epsilon)} \setminus B(1+t)$ and therefore has size

$$v^n(r, t) w^{N+1-n}(r, t) \lesssim \frac{L^n \epsilon^{-n\alpha}}{r^{N+1}} \|w\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N+1-n}.$$

In the same way as above we then obtain a bound of

$$\|(5.15)\|_{\langle x \rangle^{-1} L_{t,x}^\infty} \lesssim L^n \epsilon^{1-n\alpha} \|w\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N+1-n}.$$

For the last term corresponding to $n = 0$, we use a different method, namely

$$\left\| \left(\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} w^{N+1}(s) ds \right) (x, t) \right\|_{L_t^\infty} \lesssim \int_{\mathbb{R}^3} \frac{1}{|x-y|} \|w(y, s)\|_{L_s^\infty}^{N+1} dy.$$

Note that for $n > 3$, by subdividing the integration domain into $|x-y| \leq \frac{|x|}{2}$ and $|x-y| \geq \frac{|x|}{2}$, we obtain

$$\left| \frac{1}{|x|} * \frac{1}{\langle x \rangle^n} \right| = \int_{\mathbb{R}^3} \frac{1}{|x-y|} \frac{1}{\langle y \rangle^n} dy \lesssim \frac{1}{\langle x \rangle}.$$

Consequently, for $N+1 > 3$

$$\left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} w^{N+1}(s) ds \right\|_{\langle x \rangle^{-1} L_{t,x}^\infty} \lesssim \|w\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N+1}.$$

We linearize equation (5.14) to

$$w_{tt} - \Delta w \pm (v + \tilde{w})^{N+1} = 0, \quad w(0) = 0, \quad w_t(0) = 0. \quad (5.16)$$

Putting everything together, since this equation has null initial data, we have obtained that

$$\|w\|_{\langle x \rangle^{-1} L_{t,x}^\infty} \lesssim L^{N+1} \epsilon^{1-(N+1)\alpha} + L \epsilon^{1-\alpha} \|\tilde{w}\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^N + \|\tilde{w}\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N+1}.$$

Assuming that $\|\tilde{w}\|_{\langle x \rangle^{-1} L_{t,x}^\infty} \leq \epsilon_0 \ll 1$, we retrieve the same for w if we assume that ϵ is small enough and that $\alpha < \frac{1}{N+1}$.

Consider two pairs w_1 and \tilde{w}_1 , respectively w_2 and \tilde{w}_2 , which fulfill (5.16). In the same manner as before we obtain that

$$\begin{aligned} \|w_1 - w_2\|_{\langle x \rangle^{-1} L_{t,x}^\infty} &\lesssim \|\tilde{w}_1 - \tilde{w}_2\|_{\langle x \rangle^{-1} L_{t,x}^\infty} (L^N \epsilon^{1-N\alpha} + L \epsilon^{1-\alpha} (\|\tilde{w}_1\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N-1} + \\ &\quad + \|\tilde{w}_2\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N-1}) + \|\tilde{w}_1\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^N + \|\tilde{w}_2\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^N). \end{aligned}$$

We obtain that the mapping $\tilde{w} \mapsto w$ is a contraction on $\{w \mid \|w\|_{\langle x \rangle^{-1} L_{t,x}^\infty} \leq \epsilon_0\}$ if ϵ_0 and ϵ are sufficiently small and if $\alpha < \frac{1}{N}$. Consequently it has a fixed point w such that $u = v + w$ is a solution to (1.1).

Since we want to obtain a dispersive solution, we shall also keep track throughout the contraction scheme of the $L_x^\infty L_t^1$ norm (in fact we can do better and we shall bound the $\langle x \rangle^{-1} L_x^\infty L_t^1$ norm). This is sufficient in view of the fact that

$$\langle x \rangle^{-1} L_{t,x}^\infty \cap L_x^\infty L_t^1 \subset L_{t,x}^{2N}.$$

Note that

$$\|w\|_{L_x^\infty L_t^1} \lesssim \|(v + \tilde{w})^{N+1}\|_{L_x^{3/2,1} L_t^1}.$$

However, this estimate is insufficient in view of the fact that v is large. Returning to our computation (5.13), we extract some better bounds. Note that (5.12) is zero for $t \leq r-2$ and that for $t \geq r-2$

$$(5.12) \lesssim L^n \epsilon^{1-n\alpha} \min \left(\frac{1}{r(1 + \max(0, \frac{t-2-r}{2}))^{n-2}}, \frac{1}{(1 + \max(0, \frac{t-2-r}{2}))^{n-1}} \right)$$

the latter by using the mean value theorem. It follows that for $n > 3$

$$\|(5.12)\|_{\langle x \rangle^{-1} L_x^\infty L_t^1} \lesssim L^n \epsilon^{1-n\alpha}.$$

We again set $n = N + 1$ and use a similar method (considering their support) to evaluate the terms (5.15) for $0 < n \leq N + 1$, resulting in

$$\|(5.15)\|_{\langle x \rangle^{-1} L_x^\infty L_t^1} \lesssim L^n \epsilon^{1-n\alpha} \|w\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N+1-n}.$$

For the remaining term of the form (5.15), in which $n = 0$, i.e. for

$$\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} w^{N+1}(s) ds, \quad (5.17)$$

we use the fact that for $N + 1 > 3$

$$\|(5.17)\|_{\langle x \rangle^{-1} L_x^\infty L_t^1} \lesssim \|\tilde{w}^{N+1}\|_{\langle x \rangle^{-N-1} L_x^\infty L_t^1} \lesssim \|\tilde{w}\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^N \|\tilde{w}\|_{\langle x \rangle^{-1} L_x^\infty L_t^1}.$$

In conclusion

$$\|w\|_{\langle x \rangle^{-1} L_x^\infty L_t^1} \lesssim L^{N+1} \epsilon^{1-(N+1)\alpha} + L \epsilon^{1-\alpha} \|\tilde{w}\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^N + \|\tilde{w}\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^N \|\tilde{w}\|_{\langle x \rangle^{-1} L_x^\infty L_t^1}.$$

Thus the mapping $w \mapsto \tilde{w}$ takes the set $\{w \mid \|w\|_{\langle x \rangle^{-1} L_x^\infty L_t^1} \leq R, \|w\|_{\langle x \rangle^{-1} L_{t,x}^\infty} \leq \epsilon_0\}$ into itself for sufficiently large R and sufficiently small ϵ and ϵ_0 .

Similarly we obtain that for two pairs w_1 and \tilde{w}_1 , respectively w_2 and \tilde{w}_2 , that satisfy (5.16),

$$\begin{aligned} \|w_1 - w_2\|_{\langle x \rangle^{-1} L_x^\infty L_t^1} &\lesssim \|\tilde{w}_1 - \tilde{w}_2\|_{\langle x \rangle^{-1} L_{t,x}^\infty} (L^N \epsilon^{1-N\alpha} + L \epsilon^{1-\alpha} (\|\tilde{w}_1\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N-1} \\ &\quad + \|\tilde{w}_2\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^{N-1})) + \|\tilde{w}_1 - \tilde{w}_2\|_{\langle x \rangle^{-1} L_x^\infty L_t^1} (\|\tilde{w}_1\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^N + \|\tilde{w}_2\|_{\langle x \rangle^{-1} L_{t,x}^\infty}^N). \end{aligned}$$

It follows that the sequence $w_0 = 0$,

$$w_{n+1} = \mp \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (v(s) + w_n(s))^{N+1} ds$$

converges in $\langle x \rangle^{-1} L_x^\infty L_t^1$ for sufficiently small ϵ and ϵ_0 (in addition to $\langle x \rangle^{-1} L_{t,x}^\infty$, which we already knew).

In particular we can take $\alpha = \frac{1}{p_c} = \frac{2}{3N} < \frac{1}{N+1}$ so that $\|u_0\|_{L^{p_c}} \sim L$ is arbitrarily large. \square

Remark 5.3. The proof works more generally whenever (u_0, u_1) are radial and outgoing, supported on $\overline{B(0, 1 + \epsilon)} \setminus B(0, 1)$ and

$$\epsilon^{1/(N+1)} \|u_0\|_{L^\infty} << 1.$$

This means that the L^{N+1} norm of u_0 must be small, but the L^p norm for $p > N + 1$ (in particular the L^{p_c} norm) can be arbitrarily high.

Of course, by rescaling we can make any L^p norm with $p \neq p_c = 3N/2$ as small or large as we like.

Remark 5.4. A more interesting case should be taking large initial data supported on the union of two thin neighboring spherical shells, with opposite signs. This should lead to improved estimates due to cancellations.

APPENDIX A. POSITIVE SOLUTIONS AND A COMPARISON PRINCIPLE

In the following, we present a "global" well-posedness result for large initial data. However, the solutions obtained can be positive or infinite.

Lemma A.1. *Suppose that (u_0, u_1) are such that $u_1 = 0$ and $(ru_0(r))' \geq 0$; then the solution to the linear wave equation in three dimensions having (u_0, u_1) as initial data is nonnegative on \mathbb{R}^{3+1} .*

Proof of Lemma A.1. Call the solution of the linear equation u . With T defined by (3.1) or (3.2), let $u = T(v)$. Then by virtue of (3.2) it suffices to prove that $v \geq 0$ and in turn this follows once we show that $v_- \geq 0$ and $v_+ \geq 0$. However, by (3.7), since $u_1 = 0$ (and hence $v_1 = 0$),

$$v_-(r) = v_+(r) = \frac{1}{2}v_0(r) = \frac{1}{4\pi}(ru_0(r))' \geq 0.$$

□

Proposition A.2. *Assume that (u_0, u_1) are radial and*
i. either outgoing according to Definition 3.5 and $u_0 \geq 0$
ii. or $u_1 = 0$ and $(ru_0(r))' \geq 0$.

Then there exists a global solution u (on $\mathbb{R}^3 \times [0, \infty)$) in the first case and on \mathbb{R}^{3+1} in the second case) to equation (1.1) with the focusing sign $-$, for $N \geq 0$, having (u_0, u_1) as initial data. Moreover, u is nonnegative or infinite.

In the first case, if $u_0 \geq v_0$, then one has for the corresponding solutions that $u \geq v$. In the second case, if $(ru_0(r))' \geq (rv_0(r))'$, then $u \geq v$.

Proof of Proposition A.2. Consider the following sequence:

$$\begin{aligned} u^0(t) &= \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1, \\ u^{n+1}(t) &= \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}|u^n(s)|^N u^n(s) ds. \end{aligned}$$

The first term in the sequence is nonnegative due to our hypothesis, by Proposition 3.8 or Lemma A.1. Inductively we see that all u^n are nonnegative, hence all the integrals are well-defined (though possibly infinite).

Furthermore, one proves by induction that the sequence $(u^n)_n$ is monotonically increasing, due to the positivity of the kernel

$$\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(x, y) = \frac{1}{4\pi t} \delta_{|x-y|=t}.$$

Let $u := \lim_{n \rightarrow \infty} u^n$ (which must exist, but can be nonnegative or infinite). By the monotone convergence theorem it follows that

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}|u(s)|^N u(s) ds,$$

i.e. u is a solution to (1.1) with initial data (u_0, u_1) .

In order to compare two solutions, note that if $u_0 \geq v_0$ (or $(ru_0(r))' \geq (rv_0(r))'$) then $u^0 \geq v^0$ and by induction $u^n \geq v^n$ for every n , so $u \geq v$. \square

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REFERENCES

- [BeGo] M. Beceanu, M. Goldberg, *Strichartz estimates and maximal operators for the wave equation in \mathbb{R}^3* , Journal of Functional Analysis (2014), Vol. 266, Issue 3, pp. 1476–1510.
- [BeLö] J. Bergh, J. Löfström, *Interpolation Spaces. An Introduction*, Springer-Verlag, 1976.
- [Bou] J. Bourgain, *Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case*, J. Amer. Math. Soc. 12 (1999), pp. 145–171.
- [Bul1] A. Bulut, *The radial defocusing energy-supercritical cubic nonlinear wave equation in \mathbb{R}^{1+5}* , preprint, arXiv:1104.2002.
- [Bul2] A. Bulut, *Global well-posedness and scattering for the defocusing energy-supercritical cubic nonlinear wave equation*, preprint, arXiv:1006.4168.
- [Bul3] A. Bulut, *The defocusing energy-supercritical cubic nonlinear wave equation in dimension five*, preprint, arXiv:1112.0629.
- [CKSTT] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on \mathbb{R}^3* , Comm. Pure Appl. Math. 57 (2004), pp. 987–1014.
- [DKM] T. Duyckaerts, C. Kenig, F. Merle, *Scattering for radial, bounded solutions of focusing supercritical wave equations*, preprint, arXiv:1208.2158.
- [EnMa] B. Engquist, A. Majda, *Absorbing boundary conditions for the numerical simulation of waves*, Math. Comp. 31 (1977), pp. 629–651.
- [GiVe] J. Ginibre, G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Func. Anal., 133 (1995), pp. 50–68.
- [GSV] J. Ginibre, A. Soffer, G. Velo, *The global Cauchy problem for the critical nonlinear wave equation*, Journal of Functional Analysis (1992), Vol. 110, Issue 1, pp. 96–130.
- [KeTa] M. Keel, T. Tao, *Endpoint Strichartz estimates*, American Journal of Mathematics (1998), Vol. 120, No. 5, pp. 955–980.
- [KeMe] C. Kenig, F. Merle, *Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation*, Acta Mathematica (2008), Vol. 201, Issue 2, pp. 147–212.
- [Ker] S. Keraani, *On the blow-up phenomenon of the critical nonlinear Schrödinger equation*, J. Funct. Anal. 235 (2006), pp. 171–192. MR2216444
- [KiVi1] R. Kilip, M. Visan, *The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions*, preprint, arXiv:1002.1756.
- [KiVi2] R. Kilip, M. Visan, *The defocusing energy-supercritical nonlinear wave equation in three space dimensions*, preprint, arXiv:1001.1761.
- [KlMa] S. Klainerman, M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math, 46 (1993), pp. 1221–1268.

- [KrSc] J. Krieger, W. Schlag, *Large global solutions for energy supercritical nonlinear wave equations on \mathbb{R}^{3+1}* , preprint, arXiv:1403.2913.
- [Roy1] T. Roy, *Scattering above energy norm of solutions of a loglog energy-supercritical Schrödinger equation with radial data*, preprint, arXiv:0911.0127.
- [Roy2] T. Roy, *Global existence of smooth solutions of a 3D loglog energy-supercritical wave equation*, preprint, arXiv:0810.5175.
- [Str] M. Struwe, *Global well-posedness of the Cauchy problem for a super-critical nonlinear wave equation in two space dimensions*, Mathematische Annalen 350.3 (2011), pp. 707–719.
- [Tao] T. Tao, *Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data*, J. Hyperbolic Diff. Eq., 4, 2007, pp. 259–266.

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